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Topological susceptibility of SU(N) gauge theories at finite temperature

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Abstract: We investigate the large-N behavior of the topological susceptibility χ in four-dimensional SU(N) gauge theories at finite temperature, and in particular across the finite-temperature transition at T_c. For this purpose, we consider the lattice formulation of the SU(N) gauge theories and perform Monte Carlo simulations for $N = 4, 6$. The results indicate that χ has a nonvanishing large-N limit for $T < T_c$, as at $T = 0$, and that the topological properties remain substantially unchanged in the low-temperature phase. On the other hand, above the deconfinement phase transition, χ shows a large suppression. The comparison between the data for $N = 4$ and $N = 6$ hints at a vanishing large-N limit for $T > T_c$.

Keywords: Gauge Field Theories, 1/N Expansion, Lattice Gauge Field Theories.

The pattern of chiral symmetry breaking for QCD with N_f light flavors at zero temperature is well understood. The symmetry group of the classical lagrangian, $\mathrm{U}(N_f)_{\mathrm{L}} \otimes \mathrm{U}(N_f)_{\mathrm{R}}$, is broken both by the anomaly and spontaneously. The spontaneous breaking of the axial subgroup $\mathrm{SU}(N_f)_A$ yields $N_f^2 - 1$ Goldstone bosons, while the anomalous breaking of the $\mathrm{U}(1)_A$ symmetry explains the heavier flavor-singlet state observed in the hadronic spectrum. (See e.g. Ref. [1].)

In the large-N limit [2], where N is the number of colors, the mass of the singlet, $m_{\eta'}$, is related to the topological susceptibility of the pure gauge theory, χ, through the well-known Witten–Veneziano (WV) formula [3, 4]:

$$F_\pi^2 m_{\eta'}^2 = 2 N_f \chi. \tag{1}$$

It is particularly interesting to study the topological susceptibility as N is varied, since, at fixed number of flavors, $1/N$ can be identified with the explicit symmetry-breaking parameter for the $\mathrm{U}(1)_A$ symmetry. Hence QCD is expected to recover the full $\mathrm{U}(N_f)_{\mathrm{L}} \otimes \mathrm{U}(N_f)_{\mathrm{R}}$ chiral symmetry as $N \to \infty$. It is then possible to show that, under very general assumptions, this chiral symmetry is spontaneously broken down to the vector subgroup $\mathrm{U}(N_f)_V$, yielding N_f^2 massless Goldstone bosons [5]. In this limit, if the topological susceptibility of the pure gauge theory does not vanish, the η' acquires a mass $m_{\eta'}^2 \sim 1/N$, i.e. it becomes a Goldstone boson whose mass squared vanishes linearly in the symmetry-breaking parameter $1/N$ as the anomaly is suppressed, see e.g. [6]. At zero temperature, numerical evidence from lattice simulations in favor of a non-vanishing large-N limit of χ, with $1/N^2$ power-law corrections, has only been obtained recently [7, 8].

As the temperature is increased, the $\mathrm{SU}(N)_A$ chiral symmetry is restored at a critical temperature T_c. The nature of this phase transition, besides its own theoretical interest, determines the dynamics of the transition from hadronic matter to a quark–gluon plasma, which is expected to take place, e.g. in heavy-ion collisions. In this respect, the effective breaking of the $\mathrm{U}(1)_A$ symmetry at finite temperature, and in particular around the transition at T_c, is of particular interest. Indeed, in the case of two light flavors, the transition may be continuous (for massless quarks) and in the $\mathrm{O}(4)$ universality class only for a sufficiently large breaking of the $\mathrm{U}(1)_A$ symmetry around T_c [9, 10]. At finite temperature the anomaly, considered as an equation between operators, remains unchanged [11], but its physical effect might be drastically different because the quantities that enter the WV formula have their own temperature dependence. In order to verify that the WV mechanism is still at work at finite temperature and in the low-temperature phase, we study the behaviour of the topological susceptibility χ as the critical temperature is approached from below, verifying that χ has a nonvanishing large-N limit, similarly to what happens at $T = 0$. As a consequence, Eq. (1) is expected to hold up to T_c. The transition between the high- and low-temperature regimes is not fully understood. At high temperatures, where instanton calculus is reliable [12], a rather different scenario

emerges. Concerning the topological properties, an interesting hypothesis has been put forward in Ref. [13]: at large N, configurations with non-trivial topological charge are exponentially suppressed in the high-temperature phase, i.e. behave as e^{-N}, so that the topological susceptibility gets rapidly suppressed in the large-N limit. In order to shed some light on this issue, we also present results from simulations above T_c.

The behavior of the topological susceptibility across the transition has already been investigated for $N = 2$ and $N = 3$ in a number of works, see e.g. Refs. [14, 15, 16, 17, 18, 19]. The main focus of this work is on the large-N behavior. For this purpose we report numerical results for SU(N) gauge theories with $N = 4, 6$. Some results for $N > 3$ have recently been reported in Ref. [19].

The behavior of the topological susceptibility at the finite-temperature deconfinement phase transition is studied in detail for SU(N) gauge theories with $N = 4, 6$, exploiting their lattice formulation given by the action

$$S = -N\gamma \sum_{x,\mu>\nu} \mathrm{Tr}\left[U_\mu(x)U_\nu(x+\mu)U_\mu^\dagger(x+\nu)U_\nu^\dagger(x) + \text{h.c.}\right], \tag{2}$$

where $U_\mu(x) \in \mathrm{SU}(N)$ are link variables. In order to study the large-N limit, it is convenient to replace the more familiar coupling β by $\gamma = \beta/2N^2$, which is the inverse of the 't Hooft coupling $\lambda = g^2 N$. In order to study the theory at finite temperature, we perform Monte Carlo simulation on asymmetric lattices. The gauge configurations are generated using a mixture of microcanonical and heat-bath updating algorithms (see Ref. [20, 21] for details). In our simulations we consider different time extensions $L_t = 6, 8$ and constant aspect ratio $L_s/L_t = 4$. These values of L_t and L_s should be sufficiently large to obtain results with small scaling and finite-size corrections. The physical temperature is given as a function of the lattice spacing and of the lattice time extension, $T = 1/a(\gamma)L_t$. Previous investigations have already shown the existence of a finite-temperature phase transition, which is first order for $N \geq 3$, see e.g. Ref. [22]. For each value of L_t, γ is tuned so as to explore an interval around the critical temperature T_c; the corresponding critical value of the coupling is denoted by γ_c. In this work the critical couplings obtained in Ref. [22] are used, i.e. $\gamma_c(L_t = 6) = 0.33717(2)$ and $\gamma_c(L_t = 8) = 0.34640(7)$ for $N = 4$, and $\gamma_c(L_t = 6) = 0.34508(5)$ for $N = 6$. When zero-temperature quantities are needed, we perform the corresponding computation on symmetric lattices at the required values of the coupling. A summary of our runs is presented in Tables 1 and 2, for $N = 4$ and $N = 6$ respectively.

The range of γ values that can actually be explored with the Wilson action is rather limited. On the one hand, γ has to be sufficiently large so that the system is actually in the weak-coupling region, i.e. beyond the first-order bulk phase transition at $\gamma = 0.339$ in the case $N = 6$, and beyond the crossover region characterized by a peak of the specific heat at $\beta = 0.325$ for $N = 4$; see Ref. [20, 21] for a more detailed

L_t	γ	$\sqrt{\sigma}_{T=0}$	$10^4\chi_{T=0}$	t	$10^4\chi$	R
6	0.335	0.296(2)	2.27(2)	−0.088(9)	2.30(4)	1.01(2)
	0.3365	0.279(3)	1.76(2)	−0.032(13)	1.80(4)	1.02(2)
	0.3369	0.275(2)	1.66(4)	−0.018(10)	1.40(4)	0.84(3)
	0.338	0.264(1)	1.416(9)	0.022(8)	*0.349(5)	*0.246(4)
	0.3395	0.258(3)	1.11(3)	0.047(14)	*0.155(3)	*0.139(5)
	0.341	0.2368(6)	0.896(10)	0.140(9)	*0.0792(13)	*0.089(2)
8	0.344	0.2160(8)	0.608(7)	−0.060(6)	0.607(13)	1.00(2)
	0.346	0.206(2)	0.48(3)	−0.015(11)	0.39(2)	0.81(7)
	0.3465	0.202(1)	0.449(15)	0.005(7)	0.198(14)	0.44(3)
	0.347	0.1981(5)	0.425(10)	0.025(6)	0.132(8)	0.31(2)
	0.348	0.1940(6)	0.357(15)	0.046(6)	0.074(3)	0.207(12)

Table 1: Finite-temperature data for the SU(4) gauge theory. $\sigma_{T=0}$ and $\chi_{T=0}$ are respectively the string tension and the topological susceptibility at $T = 0$ obtained using symmetric lattices. Data marked by an asterisk are subject to an uncontrolled systematic error due to the fact that no clear plateau was observed in the cooling procedure to determine χ.

discussion of this point. On the other hand, as γ is increased, two types of difficulties arise. First, as the lattice spacing becomes smaller, larger lattices are necessary to avoid finite-size effects. In practice, we always try to use values of γ such that the spatial extent satisfies $L_s\sqrt{\sigma} \geq 3$, where σ is the zero-temperature string tension, as suggested by previous investigations [8, 20, 21, 22]. The second obstacle is the increase of the autocorrelation time of the topological modes as the continuum limit is approached [7, 23]. Such a severe form of critical slowing down puts a stringent limit on the upper value of γ that can be efficiently simulated, at least with the currently available algorithms. Finally, since the transition is first order and rather strong in the case $N = 6$, some attention must be paid to the dependence of the Monte Carlo results on the starting configurations. In particular, close to the finite-temperature phase transition, we use hot or cold starting configurations according to the side of the transition we were investigating, to avoid hysteresis effects.

We use the following formulas and definitions for the rescaled and reduced temperatures:

$$T_r(L_t,\gamma) \equiv \frac{T}{\sqrt{\sigma}} = \frac{1}{L_t\sqrt{\sigma(\gamma)}}, \tag{3}$$

$$t(L_t,\gamma) \equiv \frac{T_r(L_t,\gamma) - T_r(L_t,\gamma_c(L_t))}{T_r(L_t,\gamma_c(L_t))} = \sqrt{\frac{\sigma(\gamma_c(L_t))}{\sigma(\gamma)}} - 1, \tag{4}$$

where, as before, σ is always computed on symmetric lattices, i.e. at $T = 0$. In order to determine the reduced temperature according to Eq. (4), we used the following

L_t	γ	$\sqrt{\sigma}_{T=0}$	$10^4\chi_{T=0}$	t	$10^4\chi$	R
6	0.344	0.2973(5)	1.79(6)	−0.071(7)	1.83(5)	1.02(4)
	0.3444	0.285(2)	1.66(7)	−0.032(10)	1.69(10)	1.02(7)
	0.3448	0.282(3)	1.50(7)	−0.021(13)	1.26(7)	0.84(6)
	0.3455	0.268(4)	1.40(12)	0.030(17)	0.096(6)	0.069(7)
	0.346	0.264(3)	1.29(7)	0.045(14)	0.061(3)	0.047(3)
	0.348	0.2535(6)	0.91(7)	0.089(8)	0.0118(8)	0.0130(13)

Table 2: Finite-temperature data for the SU(6) gauge theory. $\sigma_{T=0}$ and $\chi_{T=0}$ are respectively the string tension and the topological susceptibility at $T = 0$ obtained using symmetric lattices.

values of the string tension at γ_c: $\sqrt{\sigma} = 0.270(2)$ for $\gamma = 0.33717$ and $N = 4$ (obtained on a $12^3 \times 24$ lattice), $\sqrt{\sigma} = 0.203(1)$ for $\gamma = 0.3464$ and $N = 4$ (obtained on a $16^3 \times 32$ lattice), and $\sqrt{\sigma} = 0.276(2)$ for $\gamma = 0.34508$ and $N = 6$ (on a $12^3 \times 24$ lattice).

The topological charge Q is estimated using the cooling technique described in Ref. [7]. We compute the corresponding susceptibility $\chi \equiv \langle Q^2\rangle/V$ and the scaling ratio:

$$R(L_t, \gamma) \equiv \frac{\chi(L_t, \gamma)}{\chi(\infty, \gamma)} \,. \tag{5}$$

The results are reported in Tables 1 and 2 respectively for $N = 4$ and $N = 6$.

Before discussing the results for the topological susceptibility, let us assess the possible sources of systematic errors in the lattice computation. It is well known that topological structures may disappear during the cooling procedure. This problem is particularly severe at finite temperature, especially for small L_t, where one may not observe clear plateaux with respect to the cooling steps, see e.g. Ref. [24]. At zero temperature, a direct comparison with a fermionic estimator of the topological charge shows good agreement [25, 26], supporting the idea that the cooling method is fairly stable in this case. However, the situation at finite temperature is more difficult to control and can eventually generate a systematic error. In this respect, we expect the cooling method to perform better with increasing N. In order to minimize the possible bias due to the cooling technique, only lattices with $L_t \geq 6$ have been considered in this study. In Fig. 1 we show measurements of the topological susceptibility during cooling for $N = 4$, $L_t = 6, 8$ and $N = 6$, $L_t = 6$, and values of the couplings corresponding to the high-temperature phase, and in particular $t \approx 0.05$. Plateaux during cooling are clearly observed (from 4 to 20 cooling steps) in the simulations for $N = 6$, for $N = 4$ with $L_t = 8$, and only in the low-temperature phase in the case $N = 4$ with $L_t = 6$. This allowed us to unambiguously determine the topological susceptibility using the cooling method also at finite temperature. On

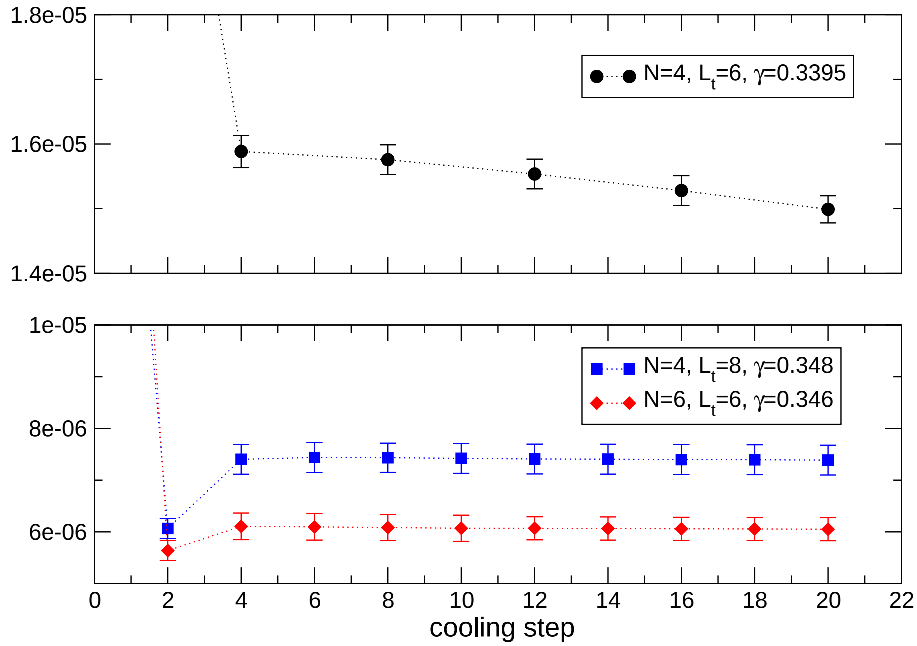

Figure 1: Topological susceptibility along the cooling process. The values of γ correspond to reduced temperatures $t \simeq 0.05$ in all cases.

the other hand, for $N = 4$ in the high-temperature regime, the time-direction size $L_t = 6$ was not sufficient to provide a clear plateau, as shown in Fig. 1 for the data corresponding to $t \approx 0.05$. Measurements based on the cooling method become rather questionable in these cases, because they do not provide an unambigous estimator of χ. In Table 1 the corresponding results are marked by an asterisk: they were obtained after 10–12 cooling steps, but they are subject to an uncontrolled systematic error, which probably leads to an underestimate of χ, and they should therefore be taken into account only with due care. For small N and L_t other estimators for Q should be used.

The data for the scaling ratio R are displayed in Fig. 2. One can immediately remark that its behavior is drastically different in the low- and high-temperature phases. In the low-temperature phase, all data for $N = 4$, $L_t = 6, 8$ and $N = 6$, $L_t = 6$ appear to lie on the same curve, showing that scaling corrections are small and also that the large-N limit is quickly approached. The ratio R remains constant and compatible with 1.0. Only close to T_c, i.e. for $T > 0.97\,T_c$, does this ratio appear to decrease. These results show that in the confined phase the topological properties remain substantially unchanged up to T_c. On the other hand, above the deconfinement phase transition, we observe a large suppression of χ. The comparison between the $N = 4$ and $N = 6$ data shows that the ratio R decreases much faster for $N = 6$, hinting at a vanishing large-N limit of R for $T > T_c$.

Numerical results supporting the same picture have also been reported recently in Ref. [19]. The numerical evidence of the topological suppression across the transition

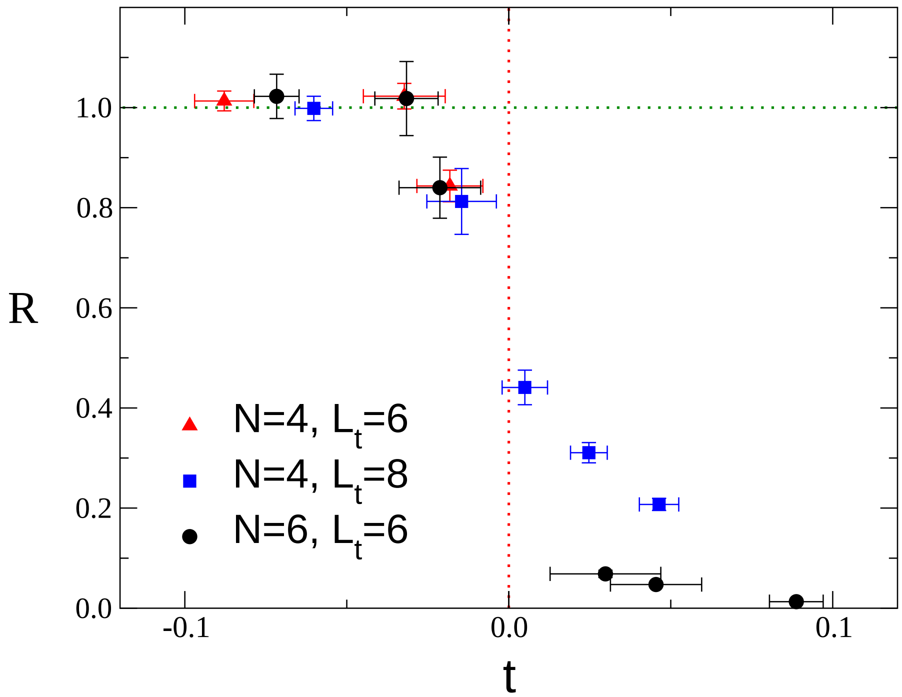

Figure 2: The ratio R as a function of the reduced temperature t.

was inferred from simulations at T_c, by monitoring the correlation of the topological charge with the Polyakov line, whose value is used to infer the actual phase of the configurations generated along the given Monte Carlo run. Therefore a more quantitative comparison with our results is not straightforward. A comparison with the results presented in Refs. [17, 18] suggests that the suppression of topological fluctuations is faster in SU(4) than it is in SU(3).

A nonvanishing topological susceptibility χ implies a nontrivial dependence on the θ term that appears in the euclidean Lagrangian as

$$\mathcal{L}_\theta = \frac{1}{4} F^a_{\mu\nu}(x) F^a_{\mu\nu}(x) - i\theta q(x) \tag{6}$$

where $q(x)$ is the topological charge density. Indeed χ is the second derivative of the free-energy density (ground-state energy) $F(\theta)$ with respect to θ at $\theta = 0$. More generally, expanding the free-energy density around $\theta = 0$, one may write

$$F(\theta) = \frac{1}{2}\chi\theta^2 \left(1 + b_2\theta^2 + b_4\theta^4 + ...\right) \tag{7}$$

The parameters of the expansion of $F(\theta)$ are related to the moments of the probability distribution $P(Q)$ of the topological charge Q in the large-volume limit. While χ is determined from the second moment $\langle Q^2 \rangle$, the coefficients b_{2i} are related to higher moments of $P(Q)$, for example

$$b_2 = -\frac{\chi_4}{12\chi}, \qquad \chi_4 = \frac{1}{V}\left[\langle Q^4 \rangle_{\theta=0} - 3\left(\langle Q^2 \rangle_{\theta=0}\right)^2\right]. \tag{8}$$

The large-volume limit of the probability distribution $P(Q)$ is Gaussian only if $b_{2i} = 0$, i.e. when $F(\theta) = \frac{1}{2}\chi\theta^2$ without higher-order corrections. A nontrivial expansion around $\theta = 0$, such as Eq. (7), reflects deviations from a simple Gaussian behavior of $P(Q)$, whose size is controlled by the coefficients b_{2i}. [1] The coefficient b_2 has been estimated in Ref. [7] for the SU(N) gauge theories with $N = 3, 4, 6$ at $T = 0$, finding very small values, i.e. $b_2 = -0.023(7)$ for $N = 3$, $b_2 = -0.013(7)$ for $N = 4$, and $b_2 = -0.01(2)$ for $N = 6$, supporting the conjecture [29, 30] $b_2 = O(1/N^2)$. Thus for $N \geq 3$ the simple Gaussian form $F(\theta) \approx \frac{1}{2}\chi\theta^2$ is expected to provide a good approximation of the dependence on θ for a relatively large range of values of θ. In order to investigate this issue at finite temperature, we have also computed b_2 in our finite-temperature Monte Carlo simulations. In the low-temperature phase the estimates of b_2 turn out to be compatible with those at $T = 0$, suggesting that $F(\theta)$ remains substantially unchanged up to $T = T_c$, with very small corrections to the Gaussian behavior. On the other hand, in the high-temperature phase the absolute value of b_2 turns out to be significantly larger, for example at $t \simeq 0.05$ we found $b_2 \simeq -0.05$ for $N = 4$ and $b_2 \simeq -0.08$ for $N = 6$, indicating larger deviations from the Gaussian behavior, although they are still moderately small.

In conclusion, the results presented in this paper suggest that the physical properties determined by the fluctuations of the topological charge, such as the Witten–Veneziano relation (1), remain substantially unchanged in the low-temperature confined phase. In particular, the topological susceptibility of the pure gauge theory has a nonvanishing large-N limit in the low-temperature phase, as at $T = 0$. On the other hand, in the high-temperature phase there is a sharp change of regime where the topological susceptibility is largely suppressed. Such suppression becomes larger with increasing N, suggesting that the topological charge vanishes above the critical temperature. Monte Carlo results seem to support the scenario presented in [13]: at large N the topological properties in the high-temperature phase, for $T > T_c$, are essentially determined by instantons from very high temperature down to T_c; the exponential suppression of instantons induces the rapid decrease of the topological activity observed in the large-N limit.

[1] An apparently contradictory result has been reported in Refs. [27, 28] for the expected large-volume probability distribution $P(Q)$, i.e. $P(Q) = (2\pi\langle Q^2\rangle)^{-1/2} e^{-\frac{Q^2}{2\langle Q^2\rangle}} [1 + O(1/V)]$, which was obtained starting from a generic expansion of $F(\theta)$ at $\theta = 0$, such as Eq. (7), and evaluating the large-volume behavior of $P(Q) = \int d\theta \exp[-iQ\theta - VF(\theta)]$ using a saddle point approximation. A Gaussian behavior in the large-volume limit [27] would contradict the assumption of a generic expansion of $F(\theta)$, and, in particular, a nonzero value of b_2, which implies a nonzero fourth moment of the large-volume $P(Q)$. The point is that the contributions considered as $O(1/V)$ corrections to the Gaussian behavior cannot be neglected in order to reproduce the correct physically relevant large-volume limit of the distribution's moments. This can be checked by computing the corrections to the saddle-point approximation.

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